III Geometric group theory – Example Sheet 2

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1. Prove that the fundamental group of the Klein bottle, $\langle a, b \mid abab^{-1} \rangle$, is quasi-isometric to the Euclidean plane $\mathbb{R}^2$. [You may use the fact that any complete, simply connected surface of constant curvature zero is isometric to $\mathbb{R}^2$.]

2. (a) Consider an isometry ϕ of a tree T that doesn't fix any points. Suppose that ϕ preserves an embedded bi-infinite line $\ell \subseteq T$. Prove that ℓ is the set of points moved a minimal distance by ϕ.
   (b) Now let T be the Cayley tree of a free group F_n. Define an *axis* for every non-trivial element of F, and prove that it coincides with the definition given in lectures for cyclically reduced elements.
   (c) Assuming that $n \geq 2$, prove that the centre of F_n is trivial.

3. Draw the Bass–Serre trees of the following graphs of groups.
   (a) $\mathbb{Z}*_1$
   (b) $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$
   (c) $\mathbb{Z}*_{\mathbb{Z}\sim 2\mathbb{Z}}$

4. Prove that the free product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is quasi-isometric to the rank-two free group F_2.

5. Give an example of a group G and an element $g \in G$ of infinite order such that the inclusion $\langle g \rangle \leq G$ is not a quasi-isometric embedding.

6. Consider an isometry ϕ of a tree T. By considering the convex hull of the points $x, \phi(x), \phi^2(x)$ for any $x \in T$, show that either ϕ fixes a point, or there is an embedded interval $[y, \phi(y)]$ such that $[y, \phi(y)] \cap [\phi(y), \phi^2(y)] = \{\phi(y)\}$. Deduce that every isometry of a tree either fixes a point or preserves a line. (As before, this line is called the *axis* of an element.)

7. Consider a finite group H acting on a tree T, and consider the function $f(x) = \max_{g \in H} d(x, gx)$ on T.
   (a) Prove that f is continuous.
   (b) Suppose that
   $$d(x, g_1 x) = d(x, g_2 x) = f(x) > 0$$
   for $g_1, g_2 \in H$. Prove that x does not lie between $g_1 x$ and $g_2 x$. By restricting attention to the convex hull of an orbit of H, deduce that H fixes a point on T.
   (c) Conclude that, if T is the Bass–Serre tree of a graph of groups with fundamental group G, then H is conjugate into a vertex group of G.

8. Consider the (2,3)-Baumslag–Solitar group $BS(2,3) = \langle a, b \mid ba^2b^{-1}a^{-3} \rangle$. Prove that there is a surjective homomorphism ϕ from $BS(2,3)$ to itself that sends a to a^2 and b to b. Show that $g = [bab^{-1}, a^2]$ is a non-trivial element in the kernel of ϕ.

9. Let Σ be the closed surface of genus two, and write $\pi_1(\Sigma) = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] \rangle$ as an amalgamated free product
$$\pi_1(\Sigma) = \langle a_1, b_1 \rangle *_{\langle [a_1, b_1] \rangle = \langle [a_2, b_2]^{-1} \rangle} \langle a_2, b_2 \rangle \,.$$
Draw a picture of this decomposition. By considering the subtrees of the Bass–Serre tree fixed by a_1 and a_2, prove that the centre of $\pi_1(\Sigma)$ is trivial.

10. Consider the presentation for *Higman's group* $H = \langle a, b, c, d \mid bab^{-1}a^{-2}, cbc^{-1}b^{-2}, dcd^{-1}c^{-2}, ada^{-1}d^{-2} \rangle$.

   (a) Prove that every finite quotient of H is trivial. [Hint: Given a non-trivial finite quotient Q, consider the smallest prime dividing the order of the image of a.]

   (b) By writing H as an amalgamated free product, prove that H is infinite.

   (c) By considering a maximal proper normal subgroup of H, show that H has a quotient that is an infinite simple group.